LITERAL EXPRESSION FOR THE MOTION OF THE MOON'S PERIGEE.

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The earlier investigators of the Lunar Theory contented themselves with giving numerical values for this quantity. The results of Clairaut, D'Alembert, Euler, Laplace and Damoiseau are of this nature. Beyond the rudest approximation, Plana was the first to give the value in a literal form. This was nearly reproduced by Pontécoulant. The salient portion, which is a function of the ratio of the month to the year, is given by these two authors to the order of the seventh power of this ratio. Delaunay, in his treatment of the subject, (*Comptes Rendus*, t. LXXIV, p. 19) has added the two following terms, viz. those which involve the eighth and ninth powers of the mentioned ratio. The correctness of these has, however, been called in question by M. Andoyer (*Annales de la Faculté des Sciences de Toulouse*, t. VI) who has also given other values for them. They may be seen in Tisserand's *Mécanique Céleste*, t. III, p. 412. I have not been able to consult M. Andoyer's memoir, and do not know what method he used in obtaining his results. The comparison I made at the end of my memoir " On the part of the motion of the Lunar Perigee, etc. (Acta Math., Vol. VIII) of Delaunay's series with my numerical value, indicated with some probability, that the newly added terms were, one or both, too large ; which corresponds with what M. Andoyer has found. In this state of matters I have thought it would not be without interest to test the validity of M. Andoyer's corrections ; and I have determined to add two more terms to the series, viz. those factored by m^{10} and m^{11}.

For this purpose, I shall employ the method of my above-mentioned memoir. There the computation was carried out in the numerical fashion, here it is proposed to give algebraic developments. It is there shown that the determination of the lunar inequalities of the type of the evection and the motion of the perigee depend on, at least as far as a first approximation is concerned, the integration of the linear differential equation of the second order

$$D^2\mathrm{w} = \Theta\mathrm{w}\,,$$

where w is the unknown, Θ a periodic function of double the mean angular distance of the Moon from the Sun, involving only cosines, and D is an operator such that $D(a\zeta^\nu) = \nu a\zeta^\nu$, ζ being the trigonometrical exponential corresponding to the mentioned mean angular distance. The motion of the perigee depends solely on the coefficients of Θ, and these can be found when

we know the coefficients of the inequalities of the type of the variation. I have given the latter in a literal form to the 9th order inclusive (Amer. Jour. Math., Vol. I, pp. 142–143).

We adopt a moving system of rectangular coordinates, the origin being at the centre of the Earth, the axis of x constantly passing through the centre of the Sun, and, in place of x and y, we use the imaginary coordinates

$$u = x + y\sqrt{-1}\,, \quad s = x - y\sqrt{-1}\,.$$

Then

$$u = \Sigma \mathrm{a}_i \zeta^{2i+1}\,, \quad s = \Sigma \mathrm{a}_i \zeta^{-2i-1}\,,$$

where, in the summation, i receives all integral values from $-\infty$ to $+\infty$, zero included, and the a_i are constants, being each equivalent to the same constant multiplied by a function of the ratio of the month to the year, which is of the $|2i|$th order with respect to this parameter. For simplicity in writing, then, we assume that the value of a_0 is unity; consequently, as written, a_i always denotes $\frac{\mathrm{a}_i}{\mathrm{a}_0}$.

In pushing the development of Θ to the degree of approximation we desire, the values of the a_i given (Amer. Jour. Math., Vol. I, pp. 142–143) generally suffice; but it will be perceived from the approximate expression for Θ (Motion of Lunar Perigee, p. 13) that it will be necessary for the determination of c, the ratio of the anomalistic to the synodic month, to the 11th order inclusive, that we should know the term factored by m^{10} in $\mathrm{a}_1 + \mathrm{a}_{-1}$; it is not necessary, however, that a_1 and a_{-1} separately should be known to this degree of approximation. Hence, we now proceed to obtain this term. From the equations given (Am. Jour. Math., Vol. I, p. 137) by neglecting all terms whose order exceeds the 10th we derive

$$\mathrm{a}_1 + \mathrm{a}_{-1} = -\frac{3\,(2+\mathrm{m})\,\mathrm{m}^2}{6 - 4\mathrm{m} + \mathrm{m}^2}(1 + 2\,\mathrm{a}_1\mathrm{a}_{-1}) - \frac{3\,(1-\mathrm{m})\,\mathrm{m}^2}{6 - 4\mathrm{m} + \mathrm{m}^2}(\mathrm{a}_{-1}{}^2 + 2\mathrm{a}_{-2} + 2\mathrm{a}_1\mathrm{a}_{-3})$$
$$- \frac{22 - 4\mathrm{m} + \mathrm{m}^2}{6 - 4\mathrm{m} + \mathrm{m}^2}(\mathrm{a}_1\mathrm{a}_2 + \mathrm{a}_{-1}\mathrm{a}_{-2}) - 9\,\mathrm{a}_2\mathrm{a}_3\,.$$

By substituting in the right member of this the values of the a_i in powers of m it is found that the term in m^{10} in $\mathrm{a}_1 + \mathrm{a}_{-1}$ is

$$+\frac{1605921808447}{2^{18}\,.\,3^8\,.\,5^3}\,\mathrm{m}^{10}\,.$$

We have

$$\Theta = -\left[\frac{\chi}{r^3} + \mathrm{m}^2\right] + 2\left[D\log\sqrt{\frac{Du}{Ds}} + \mathrm{m}\right]^2 - [D\log\sqrt{DuDs}]^2$$
$$- D^2\,[\log\sqrt{DuDs}]\,.$$

The first term of this can be developed from the formula

$$\frac{\varkappa}{r^3} + m^2 = \frac{D^2u + 2mDu + \frac{3}{2}m^2s}{u} + \frac{5}{2}m^2 = 1 + 2m + \frac{5}{2}m^2 + \Sigma \,.\, R_i\zeta^{2i},$$

where we have $R_{-i} = R_i$. The equations which determine the values of the R_i, R_0 with an error of the 12th order, R_1 with one of the 11th order, R_2 with one of the 9th order, and R_3 with one of the 7th order, are

$$R_0 + (a_1 + a_{-1})\, R_1 + (a_2 + a_{-2})\, R_2 = \frac{3}{2}m^2a_{-1}\,,$$

$$a_{-1}R_0 + (1 + a_{-2})\, R_1 + (a_1 + a_{-3})\, R_2 + a_2R_3 = -\,4ma_{-1} + \frac{3}{2}m^2\,,$$

$$a_{-2}R_0 + (a_{-1} + a_{-3})\, R_1 + R_2 + a_1R_3 = 8\,(1 - m)\, a_{-2} + \frac{3}{2}m^2a_1\,,$$

$$a_{-1}R_2 + R_3 = 24a_{-3} + \frac{3}{2}m^2a_2\,.$$

Solving these by successive approximations in using the known literal values of the a_i, we get

$$R_0 = -\frac{9}{2^5}m^4 + 4m^5 + \frac{34}{3}m^6 + 15m^7 + \frac{2704801}{2^{13}\,.\,3^3}m^8 + \frac{122957}{2^5\,.\,3^4\,.\,5}m^9$$

$$+ \frac{1260881}{2^9\,.\,3^4\,.\,5}m^{10} - \frac{291394307}{2^7\,.\,3^6\,.\,5^3}m^{11}\,,$$

$$R_1 = \frac{3}{2}m^2 + \frac{19}{2^2}m^3 + \frac{20}{3}m^4 + \frac{43}{3^2}m^5 + \frac{18709}{2^9\,.\,3^3}m^6 + \frac{759413}{2^{10}\,.\,3^4\,.\,5}m^7$$

$$+ \frac{6675059}{2^8\,.\,3^5\,.\,5^2}m^8 - \frac{41991161}{2^7\,.\,3^6\,.\,5^3}m^9 - \frac{4528083484913}{2^{17}\,.\,3^7\,.\,5^4}m^{10}\,,$$

$$R_2 = \frac{33}{2^4}m^4 + \frac{2937}{2^6\,.\,5}m^5 + \frac{23051}{2^4\,.\,3\,.\,5^2}m^6 + \frac{97051}{2^5\,.\,5^3}m^7 + \frac{334413271}{2^{10}\,.\,3^3\,.\,5^4}m^8\,,$$

$$R_3 = \frac{1393}{2^9}m^6\,.$$

We next attend to the coefficients of $\dfrac{D^2u}{Du} = \Sigma \,.\, U_i\,\zeta^{2i}$. The formulas given for these (Motion of Lunar Perigee, p. 12) in general suffice; it is necessary, however, to push the development of U_1 and U_{-1} farther, so that terms of the 10th order may be included. Thus their equivalents read

$$U_1 = 2\,[h_1 - h_{-1}\,h_2 + h_1^2\,h_{-1} + 2h_1^3\,h_{-1}^2 - h_1^3\,h_{-2} - 3h_1\,h_{-1}^2\,h_2 + 2h_1\,h_2\,h_{-2}$$
$$+ h_{-1}^2\,h_3 - h_{-2}\,h_3]\,,$$

$$U_{-1} = -\,2\,[h_{-1} - h_1\,h_{-2} + h_{-1}^2\,h_1 + 2h_{-1}^3\,h_1^2 - h_{-1}^3\,h_2 - 3h_{-1}\,h_1^2h_{-2}$$
$$+ 2h_{-1}\,h_{-2}\,h_2 + h_1^2\,h_{-3} - h_2\,h_{-3}]\,.$$

By substituting in these equations the values of the a_i in powers of m, and making the assumption that the coefficient of m^{10} in a_1 is 0, which cannot lead us into error within the limits we set to the approximation, and putting

$$\tfrac{1}{2}(U_i + U_{-i}) = A_i\,, \quad \tfrac{1}{2}(U_i - U_{-i}) = B_i\,,$$

we get

$$A_1 = -\frac{5}{2^3}m^2 - \frac{1}{2\,.\,3}m^3 + \frac{5}{3^2}m^4 + \frac{43}{2^2\,.\,3^3}m^5 - \frac{318575}{2^{11}\,.\,3^4}m^6 - \frac{2297593}{2^8\,.\,3^5\,.\,5}m^7$$
$$-\frac{9225887}{2^5\,.\,3^6\,.\,5^2}m^8 - \frac{3471983789}{2^9\,.\,3^7\,.\,5^3}m^9 - \frac{12903700736069}{2^{19}\,.\,3^8\,.\,5^4}m^{10}\,,$$

$$B_1 = \frac{7}{2^2}m^2 + \frac{19}{2\,.\,3}m^3 + \frac{53}{2\,.\,3^2}m^4 + \frac{155}{2^2\,.\,3^3}m^5 - \frac{12941}{2^{10}\,.\,3^4}m^6 - \frac{904921}{2^9\,.\,3^5\,.\,5}m^7$$
$$-\frac{35308207}{2^9\,.\,3^6\,.\,5^2}m^8 - \frac{2190838913}{2^{10}\,.\,3^7\,.\,5^3}m^9 + \frac{29589760583167}{2^{18}\,.\,3^8\,.\,5^4}m^{10}\,,$$

$$A_2 = \frac{265}{2^7}m^4 + \frac{1067}{2^5\,.\,5}m^5 + \frac{38261}{2^4\,.\,3^2\,.\,5^2}m^6 + \frac{755591}{2^6\,.\,3^2\,.\,5^3}m^7 + \frac{405840581}{2^{10}\,.\,3^4\,.\,5^4}m^8\,,$$

$$B_2 = -\frac{3}{2^2}m^4 - \frac{403}{2^4\,.\,3\,.\,5}m^5 - \frac{3773}{2^3\,.\,3^2\,.\,5^2}m^6 - \frac{246139}{2^5\,.\,3^3\,.\,5^3}m^7 - \frac{1077852389}{2^{12}\,.\,3^4\,.\,5^4}m^8,$$

$$A_3 = -\frac{1677}{2^{11}}m^6\,,$$

$$B_3 = \frac{2431}{2^{10}}m^6\,.$$

The coefficients of the function Θ have the following equivalents:

$$\Theta_0 = 1 + 2m - \tfrac{1}{2}m^2 + 4(A_1^2 + A_2^2) + 2(B_1^2 + B_2^2) - R_0\,,$$
$$\Theta_1 = 4(1+m)A_1 - 2B_1 + 4(A_1A_2 + A_2A_3) + 2(B_1B_2 + B_2B_3) - R_1\,,$$
$$\Theta_2 = 4(1+m)A_2 - 4B_2 + 2(A_1^2 + 2A_1A_3) - B_1^2 + 2B_1B_3 - R_2\,,$$
$$\Theta_3 = 4A_3 - 6B_3 + 4A_1A_2 - 2B_1B_2 - R_3\,.$$

When the expressions in powers of m for the quantities A, B, and R are substituted in the preceding equations, the results are

$$\Theta_0 = 1 + 2m - \tfrac{1}{2}m^2 + \frac{255}{2^5}m^4 + 19m^5 + \frac{80}{3}m^6 + \frac{533}{2\,.\,3^2}m^7 + \frac{11230225}{2^{13}\,.\,3^3}m^8$$
$$+\frac{1576037}{2^7\,.\,3^4}m^9 + \frac{49359583}{2^9\,.\,3^5}m^{10} + \frac{720508007}{2^8\,.\,3^6\,.\,5}m^{11}\,,$$

$$\theta_1 = -\frac{15}{2}m^2 - \frac{57}{2^2}m^3 - 11m^4 - \frac{23}{2.3.}m^5 - \frac{68803}{2^9.3^2}m^6 - \frac{1792417}{2^{10}.3^3}m^7$$

$$-\frac{7172183}{2^7.3^4.5}m^8 - \frac{596404499}{2^9.3^5.5^2}m^9 - \frac{2641291011773}{2^{17}.3^6.5^3}m^{10},$$

$$\theta_2 = \frac{111}{2^4}m^4 + \frac{1397}{2^6}m^5 + \frac{8807}{2^4.3.5}m^6 + \frac{319003}{2^5.3^2.5^2}m^7 + \frac{252382507}{2^{10}.3^3.5^3}m^8,$$

$$\theta_3 = -\frac{11669}{2^9}m^6.$$

We employ now the system of equations given (Motion of the Lunar Perigee, p. 14) and for brevity of notation put $[i]$ for $(c+i)^2 - \theta_0$. The equations, written to the requisite degree of approximation, are

$$\begin{array}{rrrrrrl}
[-3]\,b_{-3} & -\theta_1 b_{-2} & -\theta_2 b_{-1} & -\theta_3 b_0 & & & =0, \\
-\theta_1 b_{-3} & +[-2]\,b_{-2} & -\theta_1 b_{-1} & -\theta_2 b_0 & -\theta_3 b_1 & & =0, \\
-\theta_2 b_{-3} & -\theta_1 b_{-2} & +[-1]\,b_{-1} & -\theta_1 b_0 & -\theta_2 b_1 & -\theta_3 b_2 & =0, \\
-\theta_3 b_{-3} & -\theta_2 b_{-2} & -\theta_1 b_{-1} & +[0]\,b_0 & -\theta_1 b_1 & -\theta_2 b_2 & =0, \\
 & -\theta_3 b_{-2} & -\theta_2 b_{-1} & -\theta_1 b_0 & +[1]\,b_1 & -\theta_1 b_2 & =0, \\
 & & -\theta_3 b_{-1} & -\theta_2 b_0 & -\theta_1 b_1 & +[2]\,b_2 & =0.
\end{array}$$

That the relative degree of importance of the terms of these equations may be perceived, it may be pointed out that the diagonal line of coefficients $[-3]$, $[-2]$, ..., $[1]$, $[2]$ are all of the zero order of magnitude except $[-1]$ and $[0]$, the first of which is of the first order, and the second of the third order; but the latter we need not concern ourselves about; and θ_i is of the $2i$th order. From this it follows that, if we write the quantities b in the order b_{-1}, b_1, b_{-2}, b_2, b_{-3}, leaving out b_0, which is an arbitrary quantity, the first is of the first order, and every succeeding one an order higher, so that b_{-3} is of the fifth order. In order to have the equation determining c, it is necessary to eliminate the 5 mentioned b's from the group of equations. The readiest method of accomplishing this is to proceed by successive approximations using formulas of recursion. To attain the desired degree of accuracy, three approximations are necessary, each of which will give three terms in powers of m in the value of each b involved. When the values of the five b's have been obtained and substituted in the middle equation, after the rejection of the useless factor

b_0, we have the following equation serving for the determination of c :

$$
\begin{aligned}
[0] &- \left[\frac{1}{[-1]} + \frac{1}{[1]}\right]\theta_1^2 - \left[\frac{1}{[-1]^2[-2]} + \frac{1}{[1]^2[2]}\right]\theta_1^4 \\
&- 2\left[\frac{1}{[-1][1]} + \frac{1}{[-1][-2]} + \frac{1}{[1][2]}\right]\theta_1^2\theta_2 \\
&- \left[\frac{1}{[-2]} + \frac{1}{[2]}\right]\theta_2^2 - \frac{1}{[-1]^2}\left[\frac{1}{[-1][-2]^2} + \frac{1}{[-2]^2[-3]}\right]\theta_1^6 \\
&- 2\left[\frac{1}{[-1]^2[-2]^2} + \frac{1}{[-1]^2[1][-2]} + \frac{1}{[-1][1]^2[2]}\right. \\
&\qquad \left. + \frac{1}{[-1]^2[-2][-3]} + \frac{1}{[-1][-2]^2[-3]}\right]\theta_1^4\theta_2 \\
&- \left[\frac{1}{[-1][-2]^2} + \frac{1}{[-1]^2[1]} + \frac{1}{[-1][1]^2} + \frac{1}{[-1]^2[-3]}\right. \\
&\qquad \left. + \frac{2}{[-1][1][-2]} + \frac{2}{[-1][1][2]} + \frac{2}{[-1][-2][-3]}\right]\theta_1^2\theta_2^2 \\
&- 2\left[\frac{1}{[-1][-2][-3]} + \frac{1}{[-1][1][-2]} + \frac{1}{[-1][1][2]}\right]\theta_1^3\theta_3 \\
&- 2\left[\frac{1}{[-1][2]} + \frac{1}{[-1][-3]}\right]\theta_1\theta_2\theta_3 = 0 .
\end{aligned}
$$

From this all terms unnecessary to the desired degree of approximation have been rejected.

It appears desirable to give some details as to the treatment of the foregoing equation. First we form the various products of the θ involved ; each is limited to the terms needed for the degree of approximation wished.

$$
\begin{aligned}
\theta_1^2 = {} & \frac{225}{2^2}m^4 + \frac{855}{2^2}m^5 + \frac{5889}{2^4}m^6 + 371m^7 + \frac{697679}{2^9 . 3}m^8 + \frac{853817}{2^6 . 3^2}m^9 \\
& + \frac{235899233}{2^{11} . 3^3}m^{10} + \frac{1733519201}{2^9 . 3^4 . 5}m^{11} + \frac{19979134939549}{2^{18} . 3^5 . 5^2}m^{12} ,
\end{aligned}
$$

$$
\begin{aligned}
\theta_1^4 = {} & \frac{50625}{2^4}m^8 + \frac{192375}{2^3}m^9 + \frac{2787075}{2^5}m^{10} + \frac{6370695}{2^5}m^{11} \\
& + \frac{353456169}{2^{10}}m^{12} + \frac{649258747}{2^{10}}m^{13} ,
\end{aligned}
$$

$$\theta_1^6 = \frac{11390625}{2^6} \text{m}^{12} + \frac{129853125}{2^6} \text{m}^{13} + \frac{2868159375}{2^8} \text{m}^{14},$$

$$\theta_1^2\theta_2 = \frac{24975}{2^6} \text{m}^8 + \frac{693945}{2^8} \text{m}^9 + \frac{1188267}{2^7} \text{m}^{10} + \frac{21446525}{2^{10}} \text{m}^{11}$$

$$+ \frac{4710472379}{2^{13}.3.5} \text{m}^{12},$$

$$\theta_2^2 = \frac{12321}{2^8} \text{m}^8 + \frac{155067}{2^9} \text{m}^9 + \frac{20185533}{2^{12}.5} \text{m}^{10} + \frac{85123117}{2^9.3.5^2} \text{m}^{11},$$

$$\theta_1^4\theta_2 = \frac{5619375}{2^8} \text{m}^{12} + \frac{241552125}{2^{10}} \text{m}^{13}, \quad \theta_1^2\theta_2^2 = \frac{2772225}{2^{10}} \text{m}^{12} + \frac{55958985}{2^{11}} \text{m}^{13},$$

$$\theta_1^3\theta_3 = \frac{39382875}{2^{12}} \text{m}^{12}, \qquad \theta_1\theta_2\theta_3 = \frac{19428885}{2^{14}} \text{m}^{12}.$$

In the next place, by neglecting quantities of the 7th order, the equation may be written

$$[0] - \left[\frac{1}{[-1]} + \frac{1}{[1]}\right]\theta_1^2 - \frac{\theta_1^4}{128\text{m}^2} = 0,$$

and, if we put

$$\theta'_1 = \theta_1\left[1 - \frac{3\,\theta_1^2}{64\text{m}}\right],$$

it can be given the form

$$[0]^2 + 2(\theta_0 - 1)[0] + \theta'^2_1 = 0,$$

whence is derived

$$c^2 = 1 + \sqrt{(\theta_0 - 1)^2 - \theta'^2_1}.$$

By substituting the previously given developments of θ_0 and θ_1 we get

$$c^2 = 1 + 2\text{m} - \frac{1}{2}\text{m}^2 - \frac{225}{2^4}\text{m}^3 - \frac{3135}{2^6}\text{m}^4 - \frac{139973}{2^{10}}\text{m}^5 - \frac{4611319}{2^{12}.3}\text{m}^6,$$

$$c = 1 + \text{m} - \frac{3}{2^2}\text{m}^2 - \frac{201}{2^5}\text{m}^3 - \frac{2367}{2^7}\text{m}^4 - \frac{111749}{2^{11}}\text{m}^5 - \frac{4095991}{2^{13}.3}\text{m}^6.$$

These equations are correct to the last power of m set down.

This value of c may be substituted in all the terms but the two first of the equation which determines it; and the latter is thereby reduced to a manageable form. The values of the reciprocals of the quantities denoted by the

symbols [— 1], [1], [— 2], [2], [— 3], developed to the needed degree of approximation, are

$$\frac{1}{[-1]} = -\frac{1}{2^2}\mathrm{m}^{-1} - \frac{3}{2^4} - \frac{213}{2^8}\mathrm{m} - \frac{2259}{2^{10}}\mathrm{m}^2 - \frac{70973}{2^{13}}\mathrm{m}^3 - \frac{3501259}{2^{15}.3}\mathrm{m}^4,$$

$$\frac{1}{[1]} = \frac{1}{2^3} - \frac{1}{2^4}\mathrm{m} + \frac{5}{2^6}\mathrm{m}^2 + \frac{563}{2^{10}}\mathrm{m}^3 + \frac{6119}{2^{12}}\mathrm{m}^4,$$

$$\frac{1}{[-2]} = \frac{1}{2^3} + \frac{1}{2^3}\mathrm{m} + \frac{1}{2^5}\mathrm{m}^2 - \frac{643}{2^{10}}\mathrm{m}^3 - \frac{10807}{2^{12}}\mathrm{m}^4 - \frac{532047}{2^{16}}\mathrm{m}^5,$$

$$\frac{1}{[2]} = \frac{1}{2^3.3} - \frac{1}{2^3.3^2}\mathrm{m} + \frac{13}{2^5.3^3}\mathrm{m}^2 + \frac{8557}{2^{10}.3^4}\mathrm{m}^3,$$

$$\frac{1}{[-3]} = \frac{1}{2^3.3} + \frac{1}{2^4.3}\mathrm{m}.$$

The substitution being made in the eight terms of the left-hand member of the equation, the result follows, in which, for facility of verification, we write each fraction separately and in the order in which it arises from each of the eight terms.

$$-\frac{50625}{2^{11}}\mathrm{m}^6 + \left[-\frac{1022625}{2^{12}} + \frac{24975}{2^9}\right]\mathrm{m}^7$$

$$+\left[-\frac{90037575}{2^{16}} + \frac{49095}{2^7} - \frac{4107}{2^9}\right]\mathrm{m}^8$$

$$+\left[-\frac{1462100355}{2^{18}} + \frac{54632079}{2^{15}} - \frac{57165}{2^{10}} + \frac{11390625}{2^{18}}\right]\mathrm{m}^9$$

$$+\left[-\frac{165044625741}{2^{23}} + \frac{722443913}{2^{17}} - \frac{8198151}{2^{13}.5} + \frac{705459375}{2^{20}} - \frac{13111875}{2^{17}} - \frac{924075}{2^{15}}\right]\mathrm{m}^{10}$$

$$+\left[-\frac{287970294069}{2^{22}} + \frac{87619247043}{2^{20}.5} - \frac{13794117581}{2^{17}.3^2.5^2} + \frac{92222296875}{2^{24}} - \frac{702232875}{2^{19}} - \frac{34533765}{2^{17}} + \frac{65638125}{2^{19}} + \frac{6476295}{2^{17}}\right]\mathrm{m}^{11}.$$

By summing the fractions the equation may be written

$$[0] - \left[\frac{1}{[-1]} + \frac{1}{[1]}\right] \theta_1^2 = \frac{50625}{2^{11}} m^6 + \frac{822825}{2^{12}} m^7 + \frac{65426631}{2^{16}} m^8$$
$$+ \frac{514143669}{2^{17}} m^9 + \frac{579596224169}{2^{23} . 5} m^{10} + \frac{182494574380633}{2^{24} . 3^2 . 5^2} m^{11} .$$

It will be more suitable for solution if both members are multiplied by

$$- \tfrac{1}{8} [-1] [1] = 4m - m^2 - \frac{225}{2^4} m^3 - \frac{2625}{2^6} m^4 - \frac{120517}{2^{10}} m^5 - \frac{4587389}{2^{12} . 3} m^6 .$$

The right member then becomes

$$\frac{50625}{2^9} m^7 + \frac{1595025}{2^{11}} m^8 + \frac{112880037}{2^{15}} m^9 + \frac{1422559539}{2^{17}} m^{10}$$
$$+ \frac{137176160137}{2^{20} . 5} m^{11} + \frac{47733147493393}{2^{22} . 3^2 . 5^2} m^{12} .$$

Calling this K, we have

$$c^2 = 1 - \tfrac{1}{8} \theta_1^2 + \sqrt{(\theta_0 - 1 + \tfrac{1}{8} \theta_1^2)^2 - \theta_1^2 + K + \tfrac{1}{8} (c^2 - \theta_0)^3} .$$

From the preceding expressions for c and θ_0,

$$\tfrac{1}{2} (c^2 - \theta_0)^3 = - \frac{225}{2^5} m^3 - \frac{3645}{2^7} m^4 - \frac{159429}{2^{11}} m^5 - \frac{1646333}{2^{13}} m^6 ;$$

whence

$$\tfrac{1}{8} (c^2 - \theta_0) = - \frac{11390625}{2^{15}} m^9 - \frac{553584375}{2^{17}} m^{10} - \frac{60085546875}{2^{21}} m^{11}$$
$$- \frac{1228257320625}{2^{23}} m^{12}$$

The substitutions being made in the foregoing value of c^2 and the square root extracted, we get

$$c = 1 + m - \frac{3}{2^2} m^2 - \frac{201}{2^5} m^3 - \frac{2367}{2^7} m^4 - \frac{111749}{2^{11}} m^5 - \frac{4095991}{2^{13} . 3} m^6$$
$$- \frac{332532037}{2^{16} . 3^2} m^7 - \frac{15106211789}{2^{18} . 3^3} m^8 - \frac{5975332916861}{2^{23} . 3^4} m^9$$
$$- \frac{1547804933375567}{2^{25} . 3^5 . 5} m^{10} - \frac{818293211836767367}{2^{28} . 3^6 . 5^2} m^{11} .$$

By means of the equation

$$\frac{1}{n} \frac{d\omega}{dt} = 1 - \frac{c}{1 + m} ,$$

we derive from the last result the ratio of the motion of the perigee to the mean motion in longitude, viz.:

$$\frac{1}{n}\frac{d\omega}{dt} = \frac{3}{2^2}\,\mathrm{m}^2 + \frac{177}{2^5}\,\mathrm{m}^3 + \frac{1659}{2^7}\,\mathrm{m}^4 + \frac{85205}{2^{11}}\,\mathrm{m}^5 + \frac{3073531}{2^{13}\,.\,3}\,\mathrm{m}^6$$

$$+ \frac{258767293}{2^{16}\,.\,3^2}\,\mathrm{m}^7 + \frac{12001004273}{2^{18}\,.\,3^3}\,\mathrm{m}^8 + \frac{4823236506653}{2^{23}\,.\,3^4}\,\mathrm{m}^9$$

$$+ \frac{1258410742976387}{2^{25}\,.\,3^5\,.\,5}\,\mathrm{m}^{10} + \frac{667283922679600927}{2^{28}\,.\,3^6\,.\,5^2}\,\mathrm{m}^{11}\,.$$

In this we make the substitution $\mathrm{m} = \frac{m}{1 - m}$, in which m is the parameter usually employed, and prolong the resulting series only to the 9th power of m. We obtain

$$\frac{1}{n}\frac{d\omega}{dt} = \frac{3}{2^2}\,m^2 + \frac{225}{2^5}\,m^3 + \frac{4071}{2^7}\,\mathrm{m}^4 + \frac{265493}{2^{11}}\,m^5 + \frac{12822631}{2^{13}\,.\,3}\,m^6$$

$$+ \frac{1273925965}{2^{16}\,.\,3^2}\,m^7 + \frac{66702631253}{2^{18}\,.\,3^3}\,m^8 + \frac{29726828924189}{2^{23}\,.\,3^4}\,m^9\,.$$

The two terms ending this series are identical with M. Andoyer's and there can be no doubt as to their correctness. I do not push this series to the terms involving m^{10} and m^{11}, as I think the former in terms of the parameter m is to be preferred.

The series which has just been obtained is unsatisfactory on account of its slow convergence. It would be of great utility to transform it in such a manner that the convergence should be sensibly augmented. Here it seems no course is open but to experiment. Confining our attention to parameters of the form $\frac{\mathrm{m}}{1 - \alpha\mathrm{m}}$, we may seek the value of α which brings about the greatest improvement in convergence. It is plain that the adoption of a small value for this quantity would not sensibly change the series in this respect, but as α is augmented we shall reach a value where one of the numerical coefficients vanishes; if the latter belong to a high power of m, the adjacent coefficients will be small. This is true on the assumption that the series tends to become a geometrical progression. In the present case it appears that the coefficient of m^4 is the first to vanish with augmenting α. Desiring therefore that all the coefficients may still be positive after the transformation, I adopt a value of α which is less than the value which makes the mentioned coefficient vanish. The new parameter adopted is

$$\mathsf{m} = \frac{\mathrm{m}}{1 - \frac{3}{4}\,\mathrm{m}} = \frac{m}{1 - \frac{7}{4}\,m}\,.$$

By making the denominator of α, 4, we secure the advantage that the denominators of the coefficients of the series are not augmented. With this parameter then, we have the following series:

$$\frac{1}{n}\frac{d\omega}{dt} = \frac{3}{2^2}\,\mathsf{m}^2 + \frac{141}{2^5}\,\mathsf{m}^3 + \frac{57}{2^5}\,\mathsf{m}^4 + \frac{41213}{2^{11}}\,\mathsf{m}^5 + \frac{243353}{2^{12}\,.\,3}\,\mathsf{m}^6 + \frac{84226279}{2^{16}\,.\,3^2}\,\mathsf{m}^7$$

$$+ \frac{1317113479}{2^{17}\,.\,3^3}\,\mathsf{m}^8 + \frac{1125417061277}{2^{23}\,.\,3^4}\,\mathsf{m}^9 + \frac{115179069708721}{2^{24}\,.\,3^5\,.\,5}\,\mathsf{m}^{10}$$

$$+ \frac{1065454233308527477}{2^{28}\,.\,3^6\,.\,5^2}\,\mathsf{m}^{11}\,.$$

The coefficients here diminish more rapidly than in the series proceeding according to powers of m.

Correspondent to the values of n and n' employed in my previous memoirs on the lunar theory we have $\mathsf{m} = 0.0860678013$. Substituting this in the series just given, we obtain the following result, exhibited term by term; it has been assumed that the series to start from the term involving m^{11} may be regarded as a geometrical progression having the ratio $\frac{1}{3}$,

$$\begin{aligned}\frac{1}{n}\frac{d\omega}{dt} = \;& 0.0055557498 + 0.0028092554 + 0.0000977435 + 0.0000950403 \\ & + 0.0000080501 + 0.0000049959 + 0.0000011207 + 0.0000004292 \\ & + 0.0000001260 + 0.0000000418 + \overset{\text{[remainder]}}{0.0000000209} \\ = \;& 0.0085725736\,.\end{aligned}$$

The value deduced, without resorting to any expansion in powers of a parameter, is 0.0085725730. The difference of 6 units may be attributed to the uncertainty in the estimation of the remainder or to accumulated error in forming the sum of the terms of the series.

Oct. 18, 1894.